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Distorted Risk Measures with Application to Military Capability Shortfalls

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Presentation Outline

- Motivation
- Distributions and distortion
- Examination of distortion functions
- Numerical example
- Conclusions

Motivation

- Suppose a risk's associated severity is described by a distribution

- Risk *measures* summarize distribution (e.g., mean-variance methods, VaR, conditional VaR, distorted expectation)

- Expectation dampens catastrophic outcomes – right tail may require further emphasis (risk aversion)

- Questions
  - How do distortions interact with distributions?
  - Which distortion function and parameters to select?

Distributions

- Risk (R) distribution

$$R(x) = P(X > x \mid Y = 1) \cdot (1 - p)$$

$$\begin{aligned} \text{where } X &\equiv \text{severity given undesirable outcome} \\ Y &\equiv \text{binary RV of occurrence (1=yes, 0=no)} \\ p &\equiv \text{prob of no undesirable outcome} \end{aligned}$$

- Severity (S) distribution ($p = 0$)

$$S(x) = P(X > x \mid Y = 1) \cdot$$

Distortion Effects

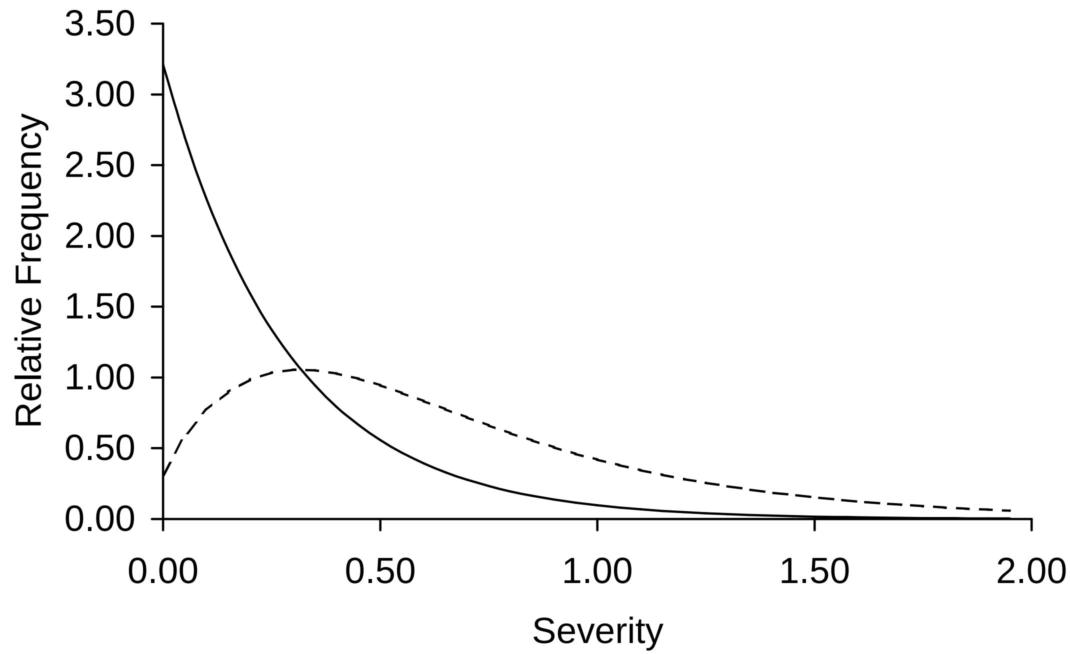

Figure 1. Distortion effects on exponential distribution.

Distortion

- Distortion function, g
  - Emphasizes worst outcomes ("pushes" density right)
  - Forms a composition, $g(S(x)) \equiv (g \circ S)(x)$
  - Is a transformation, $g : [0,1] \rightarrow [0,1]$

- Gamma-beta distortion (McLeish & Reesor, 2003)

$$g_{GB}(u) = \int_0^u K t^{a-1}(1-t)^{b-1} \exp(-t/c) dt,$$

where

$$K^{-1} = \int_0^1 t^{a-1}(1-t)^{b-1} \exp(-t/c) dt \quad \text{and} \quad u = S(x)$$

Literature Review: Distortion

- GB family – six distortions, selected parameters: gamma-beta (GB) (a,b,c), beta ($c \to \infty$), proportional hazard (PH) ($b = 1$, $c \to \infty$), dual power (DP) ($a = 1$, $c \to \infty$), gamma ($b = 1$), exponential (EX) ($a = 1$, $b = 1$) (McLeish & Reesor, 2003)

- Parameter ranges: $0 \leq a \leq 1$, $b \geq 1$, and $c \geq 0$ are sufficient to ensure *coherency* (i.e., that risks behave "reasonably") (Artzner, et al., 1997)

- Apparently no published works on appropriate choice for a distortion function or selection of associated parameters

Distribution Selections

- Unbounded distributions
  - Exponential
  - Weibull
- Bounded distributions
  - Triangular
  - Uniform

Distortion Function Selections and Effects

Selected distortion functions: PH, DP, EX, [GB]

Table 1. General distortion effects for survivor function $S(x)$.

Distortion	Parameter	$(g \circ S)(x)$
Proportional Hazard (g_{PH})	$0 < a \le 1$	$S^a(x)$
Dual Power (g_{DP})	$b \ge 1$	$1 - (1 - S(x))^b$
Exponential (g_{EX})	$0 < c < \infty$	$\frac{1-\exp(-S(x)/c)}{1-\exp(-1/c)}$
Gamma-Beta (g_{GB})	a, b, c (as above)	$\frac{\int_0^{S(x)} t^{a-1}(1-t)^{b-1}e^{-t/c}\,dt}{\int_0^1 t^{a-1}(1-t)^{b-1}e^{-t/c}\,dt}$

Distortion Parameters for Experimentation

- Background
  - New distortion measure for GB req'd (expectation N/A)
  - If ψ is undistorted median, $R_g = \frac{(g \circ S)(\psi)}{S(\psi)}$
- "Region of sensitivity" for $R_g \Rightarrow 1 \leq R_g \leq 2$ (loses track of "distance pushed")
- 3^k-factorial design for GB – "fair" analysis required each parameter have equal influence over R_g measure
- Face-centered cube: three values, equally spaced

Selected Distortion Parameter Treatments

Table 2. Selected distortion parameter treatments.

Distortion (Parameter)	Selected Value(s)		R_g (% density shift)
Proportional Hazard (a)	High	0.9	1.07 (7%)
	Mid	0.75	1.19 (19%)
	Low	0.6	1.32 (32%)
Dual Power (b)	Low	1.1	1.07 (7%)
	Mid	1.3	1.19 (19%)
	High	1.5	1.29 (29%)
Exponential (c)	High	3.6	1.07 (7%)
	Mid	2.2	1.11 (11%)
	Low	0.8	1.30 (30%)

Analytical Expectation Results

- Explicit expressions for distorted expectation risk measure
- For single-parameter distortions, numerical results attainable even when analytical expectation intractable

Table 3. Summary of risk measures, $X \sim \exp(\lambda)$.

Distortion	$\widehat{S}(x)$	$\widehat{E}[X]$
g_{PH}	$e^{-\lambda a x}$	$(\lambda a)^{-1}$
g_{DP}	$1-(1-e^{-\lambda x})^b$	$\int_0^\infty [1-(1-e^{-\lambda x})^b]\, dx$
g_{EX}	$\frac{1-\exp(-e^{-\lambda x}/c)}{1-\exp(-1/c)}$	$\int_0^\infty \frac{1-\exp(-e^{-\lambda x}/c)}{1-\exp(-1/c)}\, dx$

Weibull Distribution: Analytical Results

Table 4. Summary of risk measures, $X \sim$ Weib(β, θ).

Distortion	$\widehat{S}(x)$	$\widehat{E}[X]$
g_{PH}	$e^{a(-x/\theta)^{\beta}}$	$\frac{\theta}{\beta \sqrt[\beta]{a}} \Gamma(\frac{1}{\beta})$
g_{DP}	$1-(1-e^{(-x/\theta)^{\beta}})^{b}$	$\int_0^{\infty}[1-(1-e^{(-x/\theta)^{\beta}})^{b}]\,dx$
g_{EX}	$\frac{1-\exp(-e^{(-x/\theta)^{\beta}}/c)}{1-\exp(-1/c)}$	$\int_0^{\infty} \frac{1-\exp(-e^{(-x/\theta)^{\beta}}/c)}{1-\exp(-1/c)}\,dx$

Triangular Distribution: Analytical Results

Table 5. Summary of risk measures, $X \sim$ tria(θ_1, θ_2, m).

Distortion	$\widehat{S}(x)$	$\widehat{E}[X]$
g_{PH}	$\left(1 - \frac{(x-\theta_1)^2}{(\theta_2-\theta_1)(m-\theta_1)}\right)^a,\ \theta_1 \le x \le m$ $\left(\frac{(\theta_2-x)^2}{(\theta_2-\theta_1)(\theta_2-m)}\right)^a,\ m < x \le \theta_2$	$\int_{\theta_1}^{m} \left(1 - \frac{(x-\theta_1)^2}{(\theta_2-\theta_1)(m-\theta_1)}\right)^a dx$ $\frac{(\theta_2-m)^{a+1}}{(2a+1)(\theta_2-\theta_1)^a}$
g_{DP}	$1 - \left(\frac{(x-\theta_1)^2}{(\theta_2-\theta_1)(m-\theta_1)}\right)^b,\ \theta_1 \le x \le m$ $1 - \left(1 - \frac{(\theta_2-x)^2}{(\theta_2-\theta_1)(\theta_2-m)}\right)^b,\ m < x \le \theta_2$	$m - \theta_1 - \frac{(m-\theta_1)^{b+1}}{(\theta_2-\theta_1)^b(2b+1)}$ $\int_{m}^{\theta_2} \left[1 - \left(1 - \frac{(\theta_2-x)^2}{(\theta_2-\theta_1)(\theta_2-m)}\right)^b\right] dx$
g_{EX}	$\frac{1-\exp\left(\frac{-1}{c}+\frac{(x-\theta_1)^2}{c(\theta_2-\theta_1)(m-\theta_1)}\right)}{1-\exp(-1/c)},\ \theta_1 \le x \le m$ $\frac{1-\exp\left(\frac{-(\theta_2-x)^2}{c(\theta_2-\theta_1)(\theta_2-m)}\right)}{1-\exp(-1/c)},\ m < x \le \theta_2$	$\int_{\theta_1}^{m} \frac{1-\exp\left(\frac{-1}{c}+\frac{(x-\theta_1)^2}{c(\theta_2-\theta_1)(m-\theta_1)}\right)}{1-\exp(-1/c)} dx$ $\int_{m}^{\theta_2} \frac{1-\exp\left(\frac{-(\theta_2-x)^2}{c(\theta_2-\theta_1)(\theta_2-m)}\right)}{1-\exp(-1/c)} dx$

Uniform Distribution: Analytical Results

Table 6. Summary of risk measures, $X \sim$ unif(θ_1, θ_2).

Distortion	$\widehat{S}(x)$	$\widehat{E}[X]$
g_{PH}	$(1 - \frac{x-\theta_1}{\theta_2-\theta_1})^a$	$(\theta_2 - \theta_1)(\frac{1}{a+1})$
g_{DP}	$1 - (\frac{x-\theta_1}{\theta_2-\theta_1})^b$	$(\theta_2 - \theta_1)(\frac{b}{b+1})$
g_{EX}	$\frac{1-\exp(-(1-\frac{x-\theta_1}{\theta_2-\theta_1})/c)}{1-\exp(-1/c)}$	$(\theta_2 - \theta_1)\left(\frac{1-c+ce^{-1/c}}{1-e^{-1/c}}\right)$

Effectiveness and Efficiency

- *Effectiveness*: $K = \mu_g/\mu_0 \Rightarrow K \geq 1$
- *Efficiency*: $\frac{K}{R_g} = \frac{\texttt{change in } \mu}{\texttt{change in density}} = \frac{\Delta\mu}{\Delta\texttt{density}}$
- Importance of combined measure
  - Without it, no ability to distinguish between pairings with identical *effectiveness* ("many-to-one" mapping)
  - Every increase in R_g = additional "step" from SME recommendations $\rightarrow$ undesirable consequence

Effectiveness and Efficiency Measures

Table 7. Summary of effectiveness and efficiency measures.

Distortion →	PH			DP			EX		
Measure ↓	$a = 0.9$	$a = 0.75$	$a = 0.6$	$b = 1.1$	$b = 1.3$	$b = 1.5$	$c = 3.6$	$c = 2.2$	$c = 0.8$
Exponential(3.5), $\mu_0 = 0.285714$									
μ_g	0.318	0.382	0.476	0.304	0.336	0.366	0.306	0.319	0.379
R_g	1.072	1.189	1.319	1.067	1.188	1.293	1.069	1.113	1.303
K	1.111	1.333	1.667	1.063	1.177	1.280	1.070	1.116	1.327
K/R_g	**1.037**	**1.121**	**1.263**	0.996	0.991	0.990	**1.001**	**1.003**	**1.019**
Weibull(2,2), $\mu_0 = 1.772454$									
μ_g	1.868	2.047	2.288	1.845	1.971	2.079	1.845	1.891	2.097
R_g	1.072	1.189	1.319	1.067	1.188	1.293	1.069	1.113	1.303
K	1.054	1.155	1.291	1.041	1.112	1.173	1.041	1.067	1.183
K/R_g	**0.983**	**0.971**	**0.978**	0.976	0.936	0.907	0.973	0.958	0.908
Triangular(1,7,4), $\mu_0 = 4.000$									
μ_g	4.116	4.322	4.578	4.103	4.279	4.425	4.097	4.159	4.428
R_g	1.072	1.189	1.319	1.067	1.188	1.293	1.069	1.113	1.303
K	1.029	1.080	1.144	1.026	1.069	1.106	1.024	1.039	1.107
K/R_g	0.960	0.909	0.868	0.961	0.901	0.856	0.958	0.934	0.849
Uniform(1,7), $\mu_0 = 4.000$									
μ_g	4.158	4.429	4.750	4.143	4.391	4.600	4.139	4.227	4.609
R_g	1.072	1.189	1.319	1.067	1.188	1.293	1.069	1.113	1.303
K	1.039	1.107	1.188	1.036	1.098	1.150	1.035	1.057	1.152
K/R_g	0.969	0.931	0.900	0.971	0.924	0.889	0.968	0.949	0.885

Preferred Distortion Functions

Table 8. Preferred distortion functions.

Distribution	Low Distortion (0-10%)	Moderate Distortion (11-20%)	Heavy Distortion (21-30+%)
Exponential(3.5)	PH	PH	PH
Weibull(2,2)	PH	PH	PH
Triangular(1,7,4)	DP	EX	PH
Uniform(1,7)	DP	EX	PH

- Achieve largest possible increase in mean given a specified maximum shift in density

- Shift density by smallest amount required to achieve a specified increase in mean

Guidelines for Distortion Selection

- GB: Inability to analytically compute distorted expectation $\Rightarrow$ less appealing choice

- If SMEs suggest exponential, Weibull: PH distortion most *efficient*

- If SMEs suggest triangular, uniform: not as clear

- If additional moments are obtained from distorted distribution (e.g., σ^2), DP and EX may be preferred over PH

Results: Decision Maker Policies

Decision maker's assigned weights capture priorities

Table 9. Notional point allocations.

Capability	Assigned Weight
A	20.0
B	30.0
C	19.0
D	13.0
E	6.0
F	6.0
G	6.0
H	0.0
J	0.0

Notional Distributions

Table 10. Notional distributions.

Capability	Distribution
A	Weibull(3.5,1101)
B	Tria(1,46773,1585)
C	Unif(1,10^4)
D	Tria(1,10^4,100)
E	Weib(2.04,24.73)
F	Weib(3.08,359.1)
G	Unif(1,100)
H	Exp(0.0063)
J	Tria(1,75,3.16)

Application of Distortion

- Proposed methodology applies distortion on distribution-by-distribution basis

- Specific distortions applied objectively in accordance with guidelines previously discussed

Table 11. Selected distortion function application results.

Capability	Distribution	$(1-p)\mu_0$	R_g	Combination	$(1-p)\mu_g$
A	Weib(3.5,1101)	24.766	1.20	PH, $a = 0.735$	27.043
B	Tria(1,46773,1585)	32.239	1.30	PH, $a = 0.62$	42.641
C	Unif(1,10^4)	37.504	1.19	EX, $c = 1.3$	42.264
D	Tria(1,10^4,100)	33.670	1.13	EX, $c = 1.9$	37.189
E	Weib(2.04,24.73)	7.887	1.06	PH, $a = 0.915$	8.238
F	Weib(3.08,359.1)	9.631	1.06	PH, $a = 0.915$	9.913
G	Unif(1,100)	18.938	1.06	DP, $b = 1.09$	19.737
H	Exp(0.0063)	38.095	1.0	N/A	38.095
J	Tria(1,75,3.16)	13.193	1.0	N/A	13.193

Linear Programming Formulation

$$\texttt{Maximize} \quad \sum_{i=1}^{9} \sum_{j=1}^{6} S_i \, m_{i,j} \, x_j$$

$$\texttt{subject to} \quad \sum_{j=1}^{6} k_j \, x_j = 25$$

$$0 \leq x_j \leq 1, \;\; j = 1, 2, \ldots, 6,$$

where

S_i is risk expectation accompanying Capability i
$m_{i,j}$ denotes mitigation to Capability i by system j
x_j is "amount" of each mitigator to be purchased
k_j is cost of any *complete* mitigator j (25 unit budget)

Optimal Purchase Plan (LP Solution)

Table 12. Notional acquisition recommendations.

Risk Measure	Mitigator 1	2	3	4	5	6
None	1.0	1.0	0.2	0.0	0.0	1.0
Undistorted Expectation	1.0	1.0	0.0	0.0	0.25	1.0
Distorted Expectation	1.0	1.0	0.0	0.25	0.0	1.0

Areas for Further Study

- Computing expectation of multi-parameter distortions
- Measures other than R_g should be considered
- Study of correlation between Pearson's skewness coefficient and normalized mean
- Application of distortion functions to other distributions
- Effects of distortion on variance

Open Forum

- Comments
- Questions

Coherency

Artzner, et al. (1997)

i. Sub-additivity. $\rho(X+Y) \leq \rho(X) + \rho(Y)$

ii. Homogeneity. $\rho(t \cdot X) = t \cdot \rho(X)$

iii. Monotonicity. $\rho(X) \geq \rho(Y)$, if $X \leq Y$

iv. Risk-free condition. $\rho(X + r \cdot n) = \rho(X) - n$

Graphical Effects: Weibull

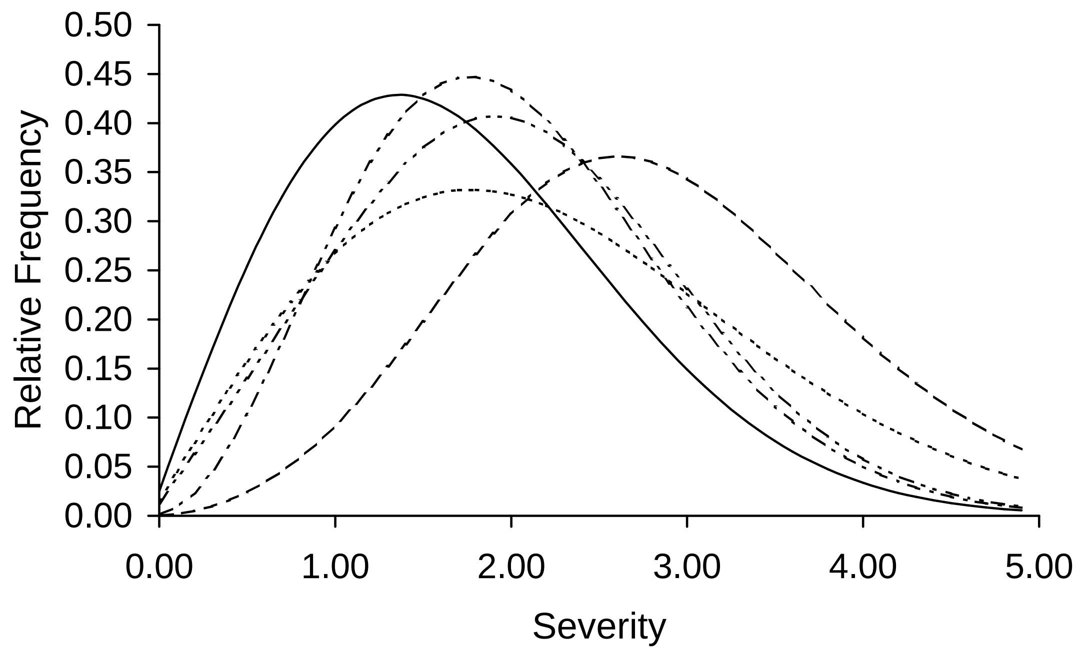

Figure 3. Relative freq density for severity, Weib(2,2) distribution, given distortion parameters $a = 0.6$, $b = 1.5$, $c = 0.8$ (solid is no distortion, $- - - -$ GB, $\cdots\cdot$ PH, $- \cdot -\cdot$ DP, $- \cdot\cdot -$ EX).

Typical μ Effects Plot (Weibull)

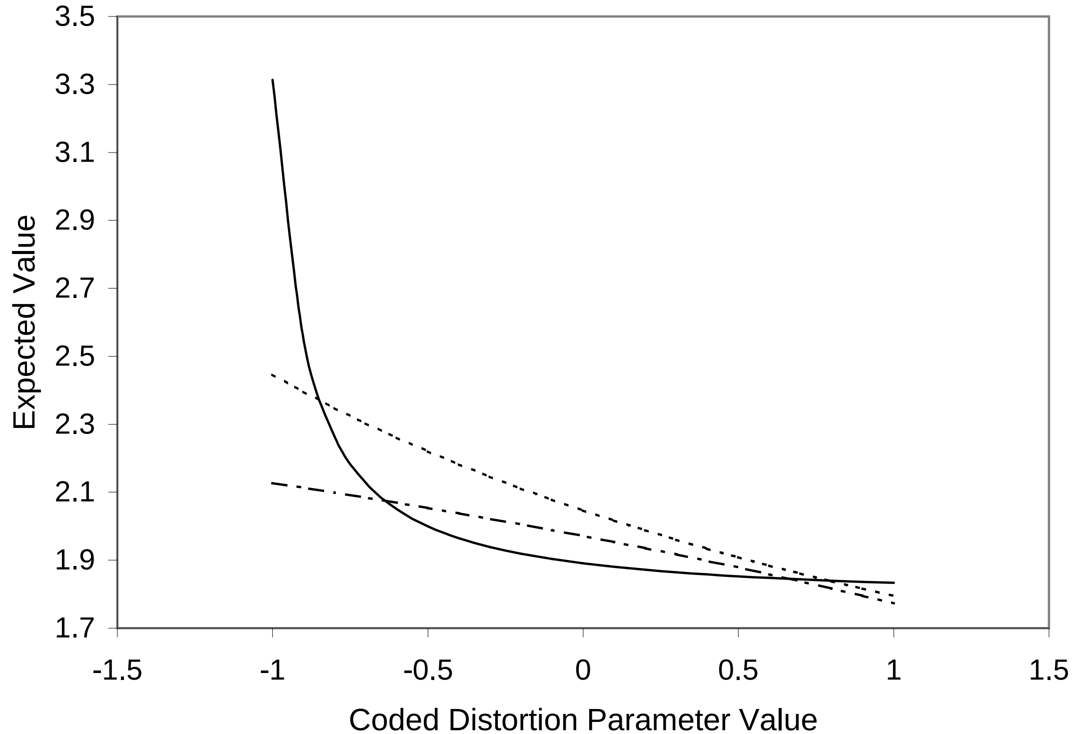

Figure 4. Expected value versus coded distortion parameters, Weib(2,2) distribution, given distortion parameter ranges $a = [0.525, 0.975]$, $b = [1, 1.6]$, $c = [0.1, 4.3]$.

Graphical Effects: Triangular

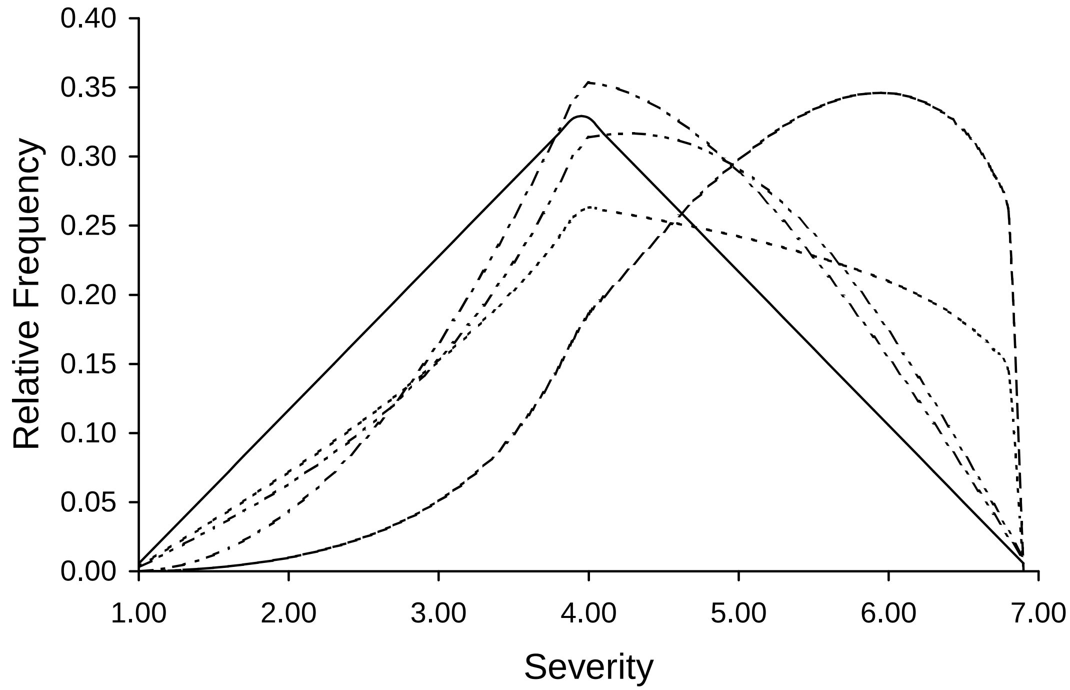

Figure 5. Relative freq density for severity, tria(1,7,4) distribution, given distortion parameters $a = 0.6$, $b = 1.5$, $c = 0.8$ (solid is no distortion, – – –– GB, $\cdots$ PH, – · –· DP, – ··– EX).

Graphical Effects: Uniform

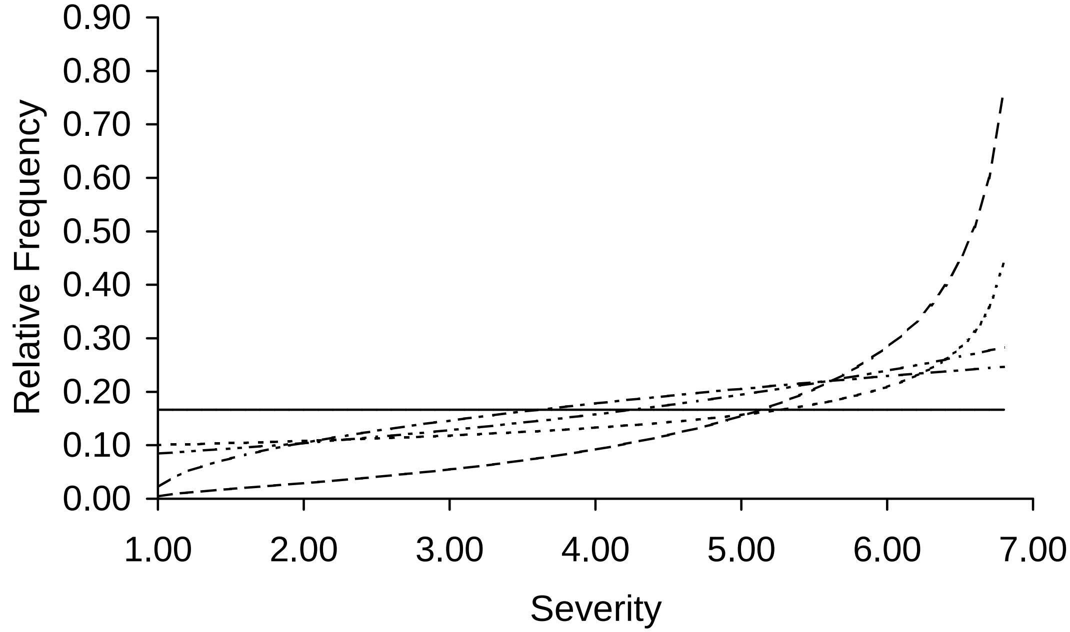

Figure 7. Relative freq density for severity, unif(1,7) distribution, given distortion parameters $a = 0.6$, $b = 1.5$, $c = 0.8$ (solid is no distortion, $- - - -$ GB, $\cdots\cdot$ PH, $- \cdot -\cdot$ DP, $- \cdot\cdot -$ EX).